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Equilibrium and non-equilibrium fluctuations in relativistic fluids

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Abstract. Starting from a causal thermodynamical approach, we study the equilibrium fluctuations of relativistic thermoviscous fluids. The results are compared with those of microscopic theory, and the transport coefficients of radiative fluids are obtained. Likewise, non-equilibrium corrections for the fluctuations of the bulk viscous pressure are evaluated for different media.

1. Introduction

So far the analysis of fluctuations of dissipative fluxes in relativistic media has received little attention in the literature, at least to our knowledge. Perhaps the most important attempt has been via the microscopic approach by Zubarev (1974). In the last few years, at the classical level, the analysis of fluctuations of dissipative fluxes and of classical variables has been unified into a simple and generalised Einstein formula for the probability of the fluctuations, namely

$$\mathrm{Pr} \sim \exp[(M/2k_{\mathrm{B}})\delta^2\eta] \tag{1.1}$$

where k_{B} is the Boltzmann constant, M the mass of the system and η a generalised specific entropy function. This generalised entropy contains as independent variables not only the conventional ones, but the dissipative fluxes as well (Casas–Vázquez and Jou 1981). In this way both kinds of quantities are considered on the same footing. The probability and the second moments of fluctuations arise immediately from a generalised Gibbs equation (Lebon *et al* 1980) in combination with Einstein's formula. The results thus obtained (Jou *et al* 1980) recover in a very direct way those of traditional approaches.

Perhaps one of the main consequences of the generalised Gibbs equation is the causal character of the ensuing transport relations, i.e. they predict finite speeds for the propagation of dissipative disturbances, in contrast with conventional approaches based on the local equilibrium hypothesis. As it can be easily understood, this property is indispensable in any relativistic treatment of fluctuations if one wishes to avoid conceptual inconsistencies.

Our first objective in this paper is to study, in § 2, the equilibrium fluctuations of relativistic thermoviscous fluids. As an application we deduce, in § 3, the phenomenological transport coefficients of a radiative fluid, which has a notable importance in both cosmological and astrophysical problems. Finally, § 4 is devoted to analysing the non-equilibrium corrections to the traditional Landau–Lifshitz formula for the bulk viscous pressure as it fluctuates around a given steady state, and we provide explicit expressions for two relativistic fluids: a mixture of leptons and photons and a gas of massive neutrinos.

As is customary, $g^{\mu\nu}$ indicates the metric tensor of signature +2, and $\Delta^{\mu\nu}$ the spatial projector $g^{\mu\nu}+u^{\mu}u^{\nu}$. The unit world velocity u^{μ} is dimensionless and fulfils the restriction $u^{\mu}\dot{u}_{\mu}=0$, where the dot denotes differentiation along the world line.

2. Equilibrium fluctuations of thermoviscous fluids

In a previous paper (Pavón *et al* 1982) we introduced the generalised Gibbs relation

$$T\dot{\eta}=\dot{\varepsilon}+p\dot{v}+v\alpha_1 q^{\mu}\dot{q}_{\mu}+v\alpha_2\Pi\dot{\Pi}+v\alpha_3 W^{\mu\nu}\dot{W}_{\mu\nu} \tag{2.1}$$

for heat conducting relativistic viscous fluids, besides the transport equations for the involved dissipative processes. These transport equations turn out to be not only of causal type but also nonlinear in the fluxes. As our purpose in this section is to study the fluctuations of dissipative fluxes around the equilibrium state, we will not consider such nonlinear terms, which are only relevant far from equilibrium. With this in mind one has

$$\alpha_1\dot{q}_{\mu}=(kcT)^{-1}q_{\mu}+(cT)^{-1}\Delta_{\mu}{}^{\nu}(T_{,\nu}+T\dot{u}_{\nu})+\alpha_1 u_{\mu}q_{\nu}\dot{u}^{\nu} \tag{2.2a}$$

$$\alpha_2\dot{\Pi}=(c\zeta)^{-1}\Pi+u^{\mu}{}_{;\mu} \tag{2.2b}$$

$$\alpha_3\dot{W}_{\mu\nu}=(2c\xi)^{-1}W_{\mu\nu}+[\![\Delta^{\lambda}{}_{(\mu}\Delta^{\rho}{}_{\nu)}u_{(\lambda;\rho)}]\!]+[\![W_{\mu\rho}u_{\nu}\dot{u}^{\rho}+u_{\mu}W_{\lambda\nu}\dot{u}^{\lambda}]\!] \tag{2.2c}$$

where the parentheses $[\![\ldots]\!]$ indicate that we must take only the traceless part of the enclosed quantity. Here q_{μ}, Π and $W_{\mu\nu}$ stand for the heat flux and the scalar and shear viscous pressure respectively, whereas the parameters α_i are defined by $\alpha_1=-\tau_1/kT$, $\alpha_2=-\tau_2/\zeta$ and $\alpha_3=-\tau_3/2\xi$. The τ_i $(i=1,2,3)$ represent the proper relaxation times of each individual dissipative process and k, ζ and ξ denote the phenomenological coefficients of heat conductivity and bulk and shear viscosities respectively.

From (1.1) and (2.1) we obtain for the probability of fluctuations of dissipative quantities the expression

$$\Pr(\delta q^{\lambda},\delta\Pi,\delta W^{\lambda\rho})\sim\exp\{(V/2k_{\mathrm{B}}T)[\alpha_1\delta q^{\mu}\delta q_{\mu}+\alpha_2(\delta\Pi)^2+\alpha_3\delta W^{\mu\nu}\delta W_{\mu\nu}]\}. \tag{2.3}$$

Let the operators $\Lambda^{\mu}{}_{\nu}[\delta q]$ and $\tilde{\Lambda}^{\mu}{}_{\nu}[\delta q]$ represent the quadratic averages $\langle\delta q^{\mu}(x^{\sigma})\delta q_{\nu}(x^{\sigma})\rangle$ and $\langle\delta q^{\mu}(x^{\sigma}+\Delta x^{\sigma})\delta q_{\nu}(x^{\sigma})\rangle$ respectively where x^{σ} and $x^{\sigma}+\Delta x^{\sigma}$ mean two arbitrary event points pertaining to the same world line. The indices attached to Λ (and $\tilde{\Lambda}$) imply the appropriate tensorial order of δq. This new notation has the advantage of compactness as compared with the usual ones used till now. Of course, the quadratic averages of the remaining fluctuating quantities admit an analogous representation.

From (2.3) the second moments in an arbitrary event point, say x^{σ}, read

$$\Lambda^{\mu}{}_{\nu}[\delta q]=(kk_{\mathrm{B}}T^2/V\tau_1)\Delta^{\mu}{}_{\nu} \tag{2.4a}$$

$$\Lambda[\delta\Pi]=\zeta k_{\mathrm{B}}T/V\tau_2 \tag{2.4b}$$

$$\Lambda^{\mu\nu}{}_{\lambda\rho}[\delta W]=(2\xi k_{\mathrm{B}}T/V\tau_3)(\Delta^{\mu}{}_{(\lambda}\Delta^{\nu}{}_{\rho)}-\tfrac{1}{3}\Delta^{\mu\nu}\Delta_{\lambda\rho}). \tag{2.4c}$$

In order to derive the correlation functions, $\tilde{\Lambda}[\delta\ldots]$, between two neighbouring event points on the same world line, say x^σ and $x^\sigma + \Delta x^\sigma$, we need the evolution equations for the fluctuations. These arise from (2.2) and in particular for the heat flux one has

$$(\delta q^\mu)^{\cdot} = -(c\tau_1)^{-1}\delta q^\mu. \tag{2.5}$$

This equation is a consequence of: (i) neglecting the relativistic temperature gradient since the temperature fluctuates more slowly than the heat flux, and (ii) setting $\dot{u}_\lambda = 0$, as the fluid is at thermodynamic equilibrium. By integrating (2.5) and combining it with (2.4*a*), we get

$$\tilde{\Lambda}^\mu{}_\nu[\delta q] = \Lambda^\mu{}_\nu[\delta q]\exp(-s/c\tau_1), \tag{2.6}$$

s being the arc length measured along the world line.

The same procedure leads to entirely similar expressions for the quantities $\tilde{\Lambda}[\delta\Pi]$ and $\tilde{\Lambda}^{\mu\nu}{}_{\lambda\rho}[\delta W]$.

By means of a very different method, Zubarev (1974) derived a set of relativistic expressions for the fluctuation-dissipation theorem corresponding to the heat conductivity and the bulk and shear viscosities. His starting point is a non-equilibrium statistical operator which describes a collectivity in a stationary state outside equilibrium. As it can be easily understood, it is of great interest for us to establish some contact with this formulation, since the ability to define microstates for non-equilibrium systems would give a firm statistical basis to the concept of non-equilibrium entropy. Therefore our objective here is to compare Zubarev's expressions with the ones deduced by us (2.4), from a phenomenological basis. This can be easily performed if an exponential decay for the fluctuations is assumed. In such a case Zubarev's expression for the bulk viscosity coefficient

$$\zeta = \frac{V}{ck_{\mathrm{B}}T}\int_0^\infty \mathrm{e}^{-\lambda s}\tilde{\Lambda}[\delta\Pi]\,\mathrm{d}s \tag{2.7}$$

reduces to (2.4*b*), and in a similar manner the indexed contracted forms of (2.4*a*) and (2.4*c*) are immediately obtainable from the corresponding Zubarev equations. Of course, the factor $\exp(-\lambda s)$ appearing in (2.7) is to ensure the convergence of the integral.

3. Phenomenological coefficients of radiative fluids

So far the calculation of phenomenological transport coefficients k, ζ and ξ of radiative fluids has been carried out, up to first order in the mean free time, through a phenomenological approach (see e.g. Weinberg 1971) or by means of a kinetic one (see e.g. Udey and Israel 1982). However, any intermediate procedure based on the theory of fluctuations has not yet been employed in that direction, at least to our knowledge. Our aim in this section is to determine such coefficients by means of the fluctuation-dissipation expressions (2.4) deduced above.

To this end, let us consider a mixture of material fluid—with very short mean free times—and radiation (photons, neutrinos, gravitons), with finite mean free time τ. The radiation is mainly responsible for the dissipation, as its mean free time is much longer than the material fluid one. Hence, any component of fluctuation of heat flux, say δq_1, reduces to $(c/\sqrt{3})\delta e_{\mathrm{rad}}$, with e_{rad} the energy density of the radiation. Then with the aid of the relation (Landau and Lifshitz 1980, pp 186, 344)

$$\Lambda[\delta e_{\mathrm{rad}}] = (k_{\mathrm{B}} C_V / V^2) T^2, \tag{3.1}$$

we have for the second moment of δq_1

$$\Lambda^1{}_1[\delta q] = (c^2 k_{\mathrm{B}} C_V / 3V) T^2 \tag{3.2}$$

$C_V (=4aVT^3)$ being the thermal capacity at constant volume of the radiation (photons for the sake of conciseness), and a the black-body constant. From (3.2) and (2.4a) we get

$$k = \tfrac{4}{3} c^2 a T^3 \tau_1. \tag{3.3}$$

To determine the coefficient of shear viscosity, ξ, we recall that any component, say W_{12}, of the viscous pressure tensor represents a moment flux, i.e. $W_{12} = e_{\mathrm{rad}} c_1 c_2 / c^2$. By using the last relation besides (3.1) and (2.4c) it follows that

$$\xi = \tfrac{4}{15} a T^4 \tau_3. \tag{3.4}$$

The determination of the bulk viscosity coefficient, ξ, requires a more subtle argument. As the radiative fluid is slightly removed from thermodynamic equilibrium a bulk viscous pressure Π, related to the equilibrium one through $\Pi = P_{\mathrm{tot}} - p$, arises. Here P_{tot} is the new total scalar pressure, and we may assume that this quantity obeys the relation $P_{\mathrm{tot}} = \nu a T^4$, ν being a numerical coefficient not depending on v or T. After these brief considerations we can write

$$\delta\Pi = \Omega \delta e_{\mathrm{rad}}, \qquad \Omega \equiv (\partial P_{\mathrm{tot}} / \partial \rho\varepsilon)_v - \tfrac{1}{3}. \tag{3.5}$$

The latter relation in conjunction with (3.1) and (2.4b) yields

$$\zeta = 4\Omega^2 a T^4 \tau_2. \tag{3.6}$$

As can be noted, if the particles making up the material fluid are very relativistic, one has $(\partial P_{\mathrm{tot}} / \partial(\rho\varepsilon))_v \to \frac{1}{3}$, and consequently $\zeta \to 0$.

Our approach has the advantage of simplicity, as compared with the rather heavy calculations involved in prior procedures. Moreover, in the expressions for k, ζ and ξ by previous authors there appears a single relaxation proper time τ, whereas in ours there appear three different times, each corresponding to a different dissipative process, which seems more natural. This result is a consequence of the Gibbs relation (2.1). If one wants to obtain explicit expressions for the τ_i it is necessary to resort to a microscopic analysis such as the one due to Straumann (1976).

4. Non-equilibrium fluctuations

In classical theory, several approaches have been proposed for the study of fluctuations around non-equilibrium steady states. Amongst them, that proposed by Jou *et al* (1982) uses the Einstein formula (1.1), whose validity is assumed for such a situation, besides a generalised Gibbs equation. In this way, these authors obtain non-equilibrium corrections to the traditional Landau–Lifshitz formulae for the fluctuations of the heat flux and the electric current.

Our concern here will be to apply the aforementioned method to derive explicit, although approximated, values for the non-equilibrium corrections of the bulk viscous pressure of some particular relativistic fluids in which the only dissipative process taking place is the one caused by the bulk viscosity. Then the corresponding generalised Gibbs equation reduces in this case to

$$\dot{\eta} = T^{-1}\dot{\varepsilon} + T^{-1}p\dot{v} - \omega\Pi\dot{\Pi}, \qquad \omega = v\tau_2/\zeta T. \tag{4.1}$$

Assuming that the system expands slowly, its mean viscous pressure $\Pi_0(\ll 1)$ is given by $\Pi_0 = -c\zeta u^{\mu}{}_{;\mu}$ and the second moment, up to second order in Π_0, can be expressed as

$$\Lambda[\delta\Pi] \simeq (k_B/M\omega)(1 - \Pi_0^2 A/\Delta), \tag{4.2}$$

Δ and A being respectively

$$\Delta = \left(\frac{\partial T^{-1}}{\partial \varepsilon}\right)_v \left(\frac{\partial T^{-1}p}{\partial v}\right)_\varepsilon - \left(\frac{\partial T^{-1}}{\partial v}\right)_\varepsilon^2 \tag{4.3a}$$

$$A = \frac{1}{\omega}\left[\left(\frac{\partial T^{-1}}{\partial \varepsilon}\right)_v \left(\frac{\partial \omega}{\partial v}\right)_\varepsilon^2 + \left(\frac{\partial T^{-1}p}{\partial v}\right)_\varepsilon \left(\frac{\partial \omega}{\partial \varepsilon}\right)_v^2 - 2\left(\frac{\partial T^{-1}}{\partial v}\right)_\varepsilon \left(\frac{\partial \omega}{\partial \varepsilon}\right)_v \left(\frac{\partial \omega}{\partial v}\right)_\varepsilon\right]. \tag{4.3b}$$

Explicit expressions for $\Lambda[\delta\Pi]$ for different types of fluids are obtainable from (4.2) if the expressions for ω, p and ε are inserted into it.

In the particular case of the fluid filling the early universe during the leptonic era—a mixture of neutrinos and electrons with temperature varying between 1.5×10^{12} K and 6×10^4 K—the energy density may be approximated by $\rho\varepsilon \simeq a'T^4$, where a' differs from a by a factor of the order of unity (see e.g. Weinberg 1972, p 536), although for practical purposes we shall not make a distinction amongst them, and $p = (\gamma - 1)\rho\varepsilon$, γ being a numerical parameter lying in the range $1 \leq \gamma \leq 2$. Concerning ζ, we will consider two possibilities. In the first place, we take for ζ the expression (3.6). Then we have

$$\Lambda[\delta\Pi] \simeq \frac{k_B}{M\omega}\left[1 - \frac{2241}{(3\gamma - 4)^3}\left(\frac{\Pi_0}{4\rho\varepsilon}\right)^2\right]. \tag{4.4}$$

In the second place, we consider ζ given by (De Groot *et al* 1980, p 231)

$$\zeta = \frac{1}{4\pi}\left(\frac{k_B T}{c\sigma(T)}\right)^4 \frac{x_\nu x_e}{432(1 + C) + 126C^2}. \tag{4.5}$$

This expression can be rewritten in a more convenient form by substituting the cross section, $\sigma(T)$, in the function of the mean free time and the particle density. Then

$$\zeta = (m^4 c^5/7020\pi\hbar^3)\tau_2. \tag{4.6}$$

Here we have considered the particle fractions x_ν and x_e being $\frac{1}{3}$ and $\frac{2}{3}$ respectively, and the parameter $C \simeq 1$. In this way one obtains

$$\Lambda[\delta\Pi] \simeq \frac{k_B}{M\omega}\left(1 - \frac{45 - \frac{3}{2}(\gamma - 1)}{2(3\gamma - 4)}\frac{\Pi_0^2}{B\varepsilon}\right) \tag{4.7}$$

where the quantity B stands for the coefficient of τ_2 on the right-hand side of (4.6).

Finally, as a third example, we consider a fluid of massive neutrinos. Recently, Calkoen and De Groot (1981) have determined its bulk viscosity coefficient, which—for $mc^2 \ll k_B T$—reads

$$\zeta \simeq \frac{k_B T}{2^6 3^2 \pi c\sigma(T)}\left(\frac{mc^2}{k_B T}\right)^4 y, \qquad y = \ln\left(\frac{k_B T}{mc^2}\right). \tag{4.8}$$

After expressing ζ as a function of τ_2 rather than as a function of the cross section, and using $7a/8$ instead of a, it is a straightforward matter to obtain

$$\Lambda[\delta\Pi] \simeq (k_B/M\omega)[1-\phi(y)4\Pi_0^2/(3\gamma-4)B\varepsilon], \tag{4.9}$$

where $B = m^4c^8/(2^63^2\pi\hbar^3c^3 \ln 10)$ and $\phi(y) = -(5y+1)^2 + \frac{3}{16}(\gamma-1)[(y+1)^2 + \frac{1}{2}(y+1)(5y+1)]$.

In writing down (4.6) and (4.9) we have made use of the simplifying assumption that τ_2 coincides with the collision time τ, such as is frequently done in the kinetic theory of gases (see e.g. Stewart 1971).

As can be noted, (4.2), and as a consequence (4.4) as well as (4.7) and (4.9), reduce to their equilibrium expression (2.4*b*) for vanishing Π_0.

Apparently, in the limit when $\gamma \to \frac{4}{3}$ the right-hand sides of equations (4.4), (4.7) and (4.9) diverge, but this is not so since $\gamma = \frac{4}{3}$ means a fluid of massless particles and hence both ζ and Π_0 vanish.

5. Conclusions

Starting from the relativistic version of extended thermodynamics, we have derived the fluctuation-dissipation expressions (2.4) for a heat conducting viscous fluid. These expressions agree with the Zubarev ones provided that an exponential decay for the fluctuations is assumed. Moreover, they allow us to derive the phenomenological transport coefficients of a radiative fluid in a quite simple manner.

The non-equilibrium corrections (4.4), (4.7) and (4.9) of the traditional Landau–Lifshitz formulae may have particular importance in the study of dissipation in Friedmannian models of the universe since, as is well known, in such models vectorial and tensorial fluxes are forbidden. Likewise, the corresponding corrections to the density fluctuations may play a non-negligible role in the initial stages of galaxy formation.

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References

Calkoen Ch J and De Groot S R 1981 *Phys. Lett.* **83A** 319
Casas-Vázquez J and Jou D 1981 *J. Phys. A: Math. Gen.* **14** 1225
De Groot S R, van Leeuwen W A and van Weert Ch G 1980 *Relativistic kinetic theory: principles and applications* (Amsterdam: North-Holland)
Jou D, Llebot J E and Casas-Vázquez J 1982 *Phys. Rev.* A **25** 508
Jou D, Rubí J M and Casas-Vázquez J 1980 *Physica* **101A** 588
Landau L and Lifshitz E M 1980 *Statistical Physics* (Oxford: Pergamon)
Lebon G, Jou D and Casas-Vázquez J 1980 *J. Phys. A: Math. Gen.* **13** 275
Pavón D, Jou D and Casas-Vázquez J 1982 *Ann. Inst. Henri Poincaré* **36A** 79
Stewart J M 1971 *Non-equilibrium relativistic kinetic theory* (New York: Springer)

Straumann N 1976 *Helv. Phys. Acta* **49** 269
Udey N and Israel W 1982 *Mon. Not. R. Astron. Soc.* **199** 1137
Weinberg S 1971 *Astrophys. J.* **168** 175
—— 1972 *Gravitation and Cosmology* (New York: Wiley)
Zubarev D N 1974 *Nonequilibrium statistical thermodynamics* (New York: Consultants Bureau) ch 4